Spectral Detection of Automorphic and Arithmetic Zeros

Mayer–Ruelle Verification, Eisenstein Scattering,

and Negative Diagnostics

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Abstract

Our methodological principle is that an operator construction for zeros of L-functions should be accepted only when it is anchored to a *rigid mathematical identity*—a theorem, not a numerical coincidence.

We verify two constructions that satisfy this criterion. *First*, the Mayer–Ruelle transfer operator for $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ detects the first four Maass cusp-form spectral parameters with up to six-digit precision and correctly discriminates their parity via the Lewis–Zagier factorisation. *Second*, the Eisenstein scattering coefficient $\varphi(s) = \xi(2s-1)/\xi(2s)$ detects the first ten Riemann zeta zeros as poles, with the functional equation $\varphi(s)\varphi(1-s) = 1$ verified to 10^{-51} and bisection refinement to 10^{-14}.

We also document the *systematic failure* of eight alternative operator constructions. The decisive diagnostic is the fixed-point analysis of $s \mapsto \zeta(s)$: the attractor $s_- \approx -0.296$ and repulsor $s_+ \approx 1.834$ govern the spectral structure of the composition operator, bearing no relation to the non-trivial zeros.

All results are deterministic and reproducible; the complete verification suite runs with two Python scripts requiring only `mpmath`.

Contents

1 Introduction

The spectral realisation of zeros of L-functions—expressing the imaginary parts t_n of the non-trivial zeros $\rho = \frac{1}{2} + it_n$ as eigenvalues or resonances of a concrete operator—has been a guiding programme in analytic number theory since Hilbert and Pólya. Connes [1] formalised this aspiration within noncommutative geometry, proposing a trace formula on the adèle class space $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$ whose "absorption spectrum" encodes the zeros of ζ. Despite its conceptual elegance, no finite-dimensional operator construction has yet been verified to produce $\{t_n\}$ as its spectrum.

This paper contributes to the programme in three ways.

(i) We provide a *complete numerical verification* of the Mayer–Ruelle transfer operator for $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ (section 3), detecting Maass cusp-form zeros with exponential convergence and correct parity discrimination.

(ii) We establish the Eisenstein scattering coefficient as a *rigid spectral observable* for Riemann zeta zeros (section 4), with full convergence analysis, negative controls, and functional-equation verification.

(iii) We document the *systematic failure* of eight alternative constructions (section 5), providing detailed diagnostics that explain *why* each approach fails. The decisive diagnostic was the fixed-point analysis of the dynamical system $s \mapsto \zeta(s)$, which identified the attractor $s_- \approx -0.296$ and repulsor $s_+ \approx 1.834$ as the true organisers of the composition operator's spectrum (section 2).

Methodological principle. Our central criterion for accepting an operator construction is the existence of a *rigid mathematical identity* connecting the operator's spectral data to an L-function—a theorem, not a numerical coincidence. Both verified results satisfy this criterion: the Mayer–Ruelle determinant equals the Selberg zeta function (up to an entire factor), and the Eisenstein scattering coefficient equals a ratio of completed zeta functions. Every construction that lacks such an identity failed our audit protocol (section 8.3).

Organisation. Section 2 develops the fixed-point diagnostic for the composition operator. Section 3 presents the Selberg–Mayer verification. Section 4 presents the Eisenstein scattering verification. Section 5 documents the negative results. Section 6 discusses connections to the Weil–Connes programme. Section 8 details the reproducibility package.

Two epistemic levels. This paper operates at two distinct levels that the reader should not conflate. The first is *theorem-backed identities*: the Mayer–Ruelle determinant identity $\det(I - \mathcal{L}_s^2) = Z_S(s) \cdot h(s)$ (section 3.1) and the Eisenstein scattering identity $\varphi(s) = \xi(2s-1)/\xi(2s)$ (theorem 4.1) are rigorous mathematical theorems. The second is *numerical realisation*: the truncation to $K \times K$ matrices, the scan resolution Δt, the bisection refinement, and the audit protocol are computational implementations subject to discretisation error and finite precision. All claims in this paper specify which level they belong to.

2 The Fixed-Point Diagnostic

The investigation began with the simplest conceivable operator: the composition operator $\mathcal{C}_\zeta \colon f \mapsto f \circ \zeta$ acting on functions on a domain $\Omega \subset \mathbb{C}$ containing part of the critical strip. If the dynamics of $s \mapsto \zeta(s)$ were sufficiently ergodic and the invariant measure concentrated near $\mathrm{Re}(s) = \frac{1}{2}$, one might hope that the transfer operator's spectrum would encode the zeros.

2.1 Fixed points of the zeta map

The map $T(s) = \zeta(s)$ has two real fixed points satisfying $\zeta(s_\pm) = s_\pm$ (see, e.g., [8]):

Definition 2.1 (Fixed points of ζ)**.**

$$s_- \approx -0.2959\ldots, \qquad |\zeta'(s_-)| \approx 0.560 < 1 \quad \text{(attractor)}, \tag{1}$$

$$s_+ \approx +1.8340\ldots, \qquad |\zeta'(s_+)| \approx 1.374 > 1 \quad \text{(repulsor)}. \tag{2}$$

The Jacobian derivative $\zeta'(s)$ at these fixed points determines the local stability: orbits near s_- are attracted to it, while orbits near s_+ are expelled. For the Perron–Frobenius operator (the adjoint of $\mathcal{C}_\zeta$), this structure dictates where the invariant measure concentrates.

2.2 Consequences for the Ulam discretisation

We discretised $\mathcal{C}_\zeta$ via the Ulam method on a grid of $N_x \times N_y$ boxes covering a domain $\mathrm{Re}(s) \in [a, b] \times \mathrm{Im}(s) \in [-c, c]$. Extensive numerical experiments (25 configurations of weight parameters (α, β) with $\alpha \in [0.3, 0.7]$, $\beta \in [1.0, 2.0]$; three domain choices; two column-closure variants; see section 5.1 for details) revealed:

(a) **The dominant eigenvalue $\lambda_1 = 1$ is an artifact.** In the "self-loop" variant (empty columns receive a self-transition), $\lambda_1 = 1$ with multiplicity 3–6 and spectral gap 0. In the open system (no artificial closure), $\lambda_1 \approx 0.53$, indicating measure escape.

(b) **The mode tracks the domain boundary, not $\mathrm{Re}(s) = 1/2$.** Three domains were tested:

Domain $\mathrm{Re}(s) \in$	Mode peak Re	Distance to 1/2
$[0.5, 2.5]$	0.53	0.03
$[0.3, 2.5]$	0.34	0.16
$[0.1, 2.5]$	0.14	0.36

The peak follows the left boundary of the computational domain.

(c) **Weight parameters are irrelevant.** Across the full (α, β) sweep, all spectral metrics vary by $< 0.5\%$. The "optimal" parameters $(\alpha, \beta) = (0.5, 1.5)$ or $(0.82, 1.18)$ produce identical results within numerical noise.

Remark 2.2 (Role of the attractor s_-). Without the attractor $s_- \approx -0.296$ lying *outside* the standard domain $\mathrm{Re}(s) \geq 0.5$, the escape of measure would have been even more severe, and the diagnostic would have been less informative. The attractor's existence means that orbits starting near the critical line are pulled toward s_- (in the left half-plane), creating a "drain" for the Perron–Frobenius measure. This is why no weight function can force the invariant measure onto $\mathrm{Re}(s) = 1/2$: the attractor lies on the wrong side of the critical line, and the dynamics are fundamentally incompatible with concentration at $\mathrm{Re}(s) = 1/2$.

Conversely, the repulsor $s_+ \approx 1.834$ explains why the unweighted operator's dominant mode peaks near $\mathrm{Re}(s) \approx 1.8$: the repulsor is a "source" for the Perron–Frobenius evolution, and measure accumulates there before escaping. The interplay between attractor and repulsor—the former draining mass from the critical strip, the latter injecting it far from $\mathrm{Re}(s) = 1/2$—constitutes the fundamental obstruction to any spectral encoding of zeros by the composition operator.

This fixed-point analysis was the turning point of the investigation. It redirected efforts from parameter tuning (which the diagnostic proved futile) toward constructions with *rigid* mathematical connections to L-functions.

3 Part I: Mayer–Ruelle Operator for the Selberg Zeta Function

3.1 Mathematical framework

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ act on the upper half-plane $\mathbb{H}$ by Möbius transformations. The Selberg zeta function for $\Gamma\backslash\mathbb{H}$ is defined for $\mathrm{Re}(s) > 1$ by the product over primitive closed geodesics $\{\gamma_0\}$ of length $\ell(\gamma_0)$:

$$Z_S(s) = \prod_{\{\gamma_0\}} \prod_{k=0}^{\infty} \bigl(1 - e^{-(s+k)\ell(\gamma_0)}\bigr). \tag{3}$$

This extends to an entire function of s, with non-trivial zeros at $s = \frac{1}{2} + ir_j$ where $\lambda_j = \frac{1}{4} + r_j^2$ are eigenvalues of the hyperbolic Laplacian Δ on $\Gamma\backslash\mathbb{H}$ (the Maass cusp forms).

3.1.1 The Gauss map and transfer operator

The continued-fraction (Gauss) map $T\colon (0,1] \to (0,1]$, $T(x) = \{1/x\}$, generates a symbolic dynamics encoding the geodesic flow on $\Gamma\backslash\mathbb{H}$. The associated transfer operator [2] is

$$(\mathcal{L}_s f)(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^{2s}}\, f\!\left(\frac{1}{x+n}\right), \qquad x \in [0,1],\ \mathrm{Re}(s) > 1/2. \tag{4}$$

Mayer [2] proved that $\mathcal{L}_s$ is nuclear of order zero on the Banach space of holomorphic functions in a disk containing $[0,1]$, and that its Fredholm determinant satisfies

$$\det(I - \mathcal{L}_s^2) = Z_S(s) \cdot h(s) \tag{5}$$

where $h(s)$ is a known entire function.

3.1.2 Lewis–Zagier factorisation

Lewis and Zagier [3] established the factorisation

$$\det(I - \mathcal{L}_s^2) = \det(I - \mathcal{L}_s) \cdot \det(I + \mathcal{L}_s), \tag{6}$$

with the two factors corresponding to *even* and *odd* Maass forms:

$$\det(I - \mathcal{L}_s) = 0 \iff \mathcal{L}_s \text{ has eigenvalue } +1 \iff \text{odd Maass cusp form at } s, \tag{7}$$

$$\det(I + \mathcal{L}_s) = 0 \iff \mathcal{L}_s \text{ has eigenvalue } -1 \iff \text{even Maass cusp form at } s. \tag{8}$$

3.2 Monomial-basis matrix elements

Proposition 3.1 (Mayer [2])**.** *In the monomial basis $\{x^k\}_{k\geq 0}$ on $[0,1]$, the operator $\mathcal{L}_s$ has matrix representation*

$$M_{jk}(s) = \frac{(-1)^j}{j!}\,(2s+k)_j\,\zeta(2s+k+j), \tag{9}$$

where $(a)_j = a(a+1)\cdots(a+j-1)$ denotes the Pochhammer symbol (rising factorial).

This representation is the key to our numerical method: it replaces the conditionally convergent infinite sum in (4) (which decays as $O(n^{-1})$ at $\mathrm{Re}(s) = \frac{1}{2}$) with evaluations of the Riemann zeta function at arguments with $\mathrm{Re}(\cdot) \geq 1 + 2\,\mathrm{Re}(s) \geq 2$. At these points ζ converges rapidly, and the matrix elements are exact (computed in arbitrary precision via `mpmath`).

The truncation to a $K \times K$ matrix $M^{(K)}(s)$ incurs an error that decreases exponentially in K, since $\mathcal{L}_s$ is nuclear of order zero.

3.3 Eigenvalue detection via inverse iteration

Standard double-precision eigenvalue algorithms (QR, ARPACK) fail for $K > 20$ because the monomial basis has Vandermonde-like conditioning $\sim K!$. We circumvent this by using *inverse iteration* targeting $\sigma \in \{+1, -1\}$:

Definition 3.2 (Inverse iteration)**.** Given a starting vector $v^{(0)}$ with $v_i^{(0)} = 1/(i+1)$ (normalised), iterate

$$w^{(n+1)} = \big(M^{(K)} - \sigma I\big)^{-1} v^{(n)}, \qquad v^{(n+1)} = w^{(n+1)}/\|w^{(n+1)}\|. \tag{10}$$

The Rayleigh quotient $\lambda^{(n)} = \sigma + 1/\big((v^{(n)})^\dagger w^{(n+1)}\big)$ converges to the eigenvalue nearest σ.

All arithmetic—matrix construction via (9), LU factorisation, and Rayleigh quotient—is performed in `mpmath` at dps $= \max(80, 2K + 20)$ decimal digits. The cost per iteration is $O(K^3)$ with $O(K)$-digit arithmetic, giving total complexity $O(K^4)$, feasible to $K \approx 60$ on a single core.

3.4 Conditioning analysis

The monomial basis $\{x^k\}$ yields a Vandermonde-like structure with theoretical conditioning $\kappa \sim K!$. At $K = 35$, this suggests $\log_{10} \kappa \approx 40$, requiring > 40 significant digits for numerical stability. Our choice dps $= \max(80, 2K + 20)$ provides a safety margin of ≈ 30 digits. Empirically, we observe that increasing precision from $2K + 20$ to $2K + 40$ digits changes all reported eigenvalue distances by $< 10^{-12}$, confirming that conditioning does not limit the observed exponential convergence.

The inverse iteration of theorem 3.2 circumvents explicit eigenvalue computation by solving $(M^{(K)} - \sigma I)w = v$ via LU factorization at precision $O(K)$ digits. The Rayleigh quotient $\lambda^{(n)} = \sigma + 1/(v^\dagger w)$ converges cubically to the nearest eigenvalue, with the ill-conditioning of $M^{(K)} - \sigma I$ manifesting only in the *residual* rather than the *eigenvalue approximation*—a standard result for inverse iteration with accurate shift $\sigma = \pm 1$.

3.5 Results

3.5.1 Trivial pole at $s = 1$

The Selberg zeta has a known trivial pole at $s = 1$, corresponding to $\det(I - \mathcal{L}_1) = 0$, i.e., eigenvalue $+1$. Table 1 confirms exponential convergence, reaching machine zero at $K \geq 55$.

Table 1: Distance $|\lambda_{\text{near}} - 1|$ at $s = 1$ (trivial zero of Z_S). The convergence rate is approximately $e^{-0.7K}$.

K	dps	$\lvert\lambda - 1\rvert$
10	80	2.38×10^{-4}
15	80	1.20×10^{-5}
20	80	2.04×10^{-7}
25	80	2.14×10^{-9}
30	80	1.01×10^{-8}
35	90	1.18×10^{-11}
45	110	3.77×10^{-15}
55	140	0 (machine zero)

3.5.2 Maass cusp forms

Table 2 presents the main Selberg result: the distance from the eigenvalue nearest the *parity-correct* target to that target, as a function of K. The known Maass spectral parameters, computed to

high precision by Booker–Strömbergsson–Venkatesh [6, Table 1] using the trace formula with rigorous error bounds, are:

$$\begin{aligned} r_1 &= 9.5336952613136\ldots \quad \text{(even)},\\ r_2 &= 12.1730083246697\ldots \quad \text{(even)},\\ r_3 &= 13.7797513518907\ldots \quad \text{(odd)},\\ r_4 &= 14.3585095182969\ldots \quad \text{(even)}. \end{aligned} \tag{11}$$

These values agree with earlier computations by Hejhal [4] and Then [7] to all digits quoted in those works. We use the 13-digit truncations above as ground truth for precision assessment.

Table 2: Distance to parity-correct target eigenvalue at the first four Maass spectral parameters. Even forms (r_1, r_2, r_4): target -1. Odd form (r_3): target $+1$. Exponential convergence is observed, with rate decreasing for larger r_j.

K	r_1=9.5337 even, -1	r_2=12.1730 even, -1	r_3=13.7798 odd, $+1$	r_4=14.3585 even, -1
10	1.26×10^{-1}	3.47×10^{-1}	6.75×10^{-1}	5.73×10^{-1}
15	2.78×10^{-2}	1.78×10^{-1}	4.87×10^{-1}	3.89×10^{-1}
18	6.47×10^{-3}	9.17×10^{-2}	1.08×10^{-1}	2.91×10^{-1}
20	9.42×10^{-4}	4.49×10^{-2}	9.92×10^{-2}	1.69×10^{-1}
25	1.90×10^{-5}	5.28×10^{-3}	3.30×10^{-2}	6.35×10^{-2}
30	6.71×10^{-6}	2.36×10^{-4}	2.22×10^{-3}	4.76×10^{-3}
35	6.21×10^{-6}	5.48×10^{-6}	1.18×10^{-4}	6.06×10^{-4}

3.5.3 Parity discrimination

The factorisation (6) predicts that even forms appear at eigenvalue -1 and odd forms at $+1$. Table 3 confirms this: at $K = 30$, the correct target gives distances $< 7 \times 10^{-3}$, while the wrong target gives $> 3.9 \times 10^{-1}$. The discrimination ratio exceeds 10^4 for r_1.

Table 3: Parity discrimination at $K = 30$: distance to both targets for each Maass zero. Bold entries indicate the parity-correct target.

Form	Parity	$\lvert\lambda - (+1)\rvert$	$\lvert\lambda - (-1)\rvert$	Status
r_1	even	7.88×10^{-1}	$\mathbf{6.71 \times 10^{-6}}$	PASS
r_2	even	3.97×10^{-1}	$\mathbf{2.36 \times 10^{-4}}$	PASS
r_3	odd	$\mathbf{2.22 \times 10^{-3}}$	9.16×10^{-1}	PASS
r_4	even	8.28×10^{-1}	$\mathbf{4.76 \times 10^{-3}}$	PASS

This structural prediction—that r_3 goes to eigenvalue $+1$ while r_1, r_2, r_4 go to -1—cannot be faked by overfitting. It is a consequence of the operator's symmetry under the involution $x \mapsto 1 - x$ and the Lewis–Zagier factorisation, and it matches the known parity of Maass forms for $\mathrm{PSL}(2, \mathbb{Z})$ [7, 6].

Remark 3.3 (Convergence at K=35 for r_3 and r_4). The parity-correct distances for r_3 and r_4 (2.2×10^{-3} and 4.8×10^{-3} respectively) are significantly larger than for r_1 and r_2 (6.7×10^{-6} and 2.4×10^{-4}). This reflects the slower convergence at higher spectral parameters, where the Pochhammer coefficients grow faster. Increasing the truncation to K=40 or K=45 would sharpen these values but requires proportionally higher `mpmath` precision (we use dps $= 2K{+}20 = 90$ at K=35). The discrimination remains unambiguous (distance to the correct target is three orders of magnitude smaller than to the incorrect one), but the absolute precision for r_3 and r_4 should be regarded as marginal.

3.5.4 Negative controls

At spectral parameters where no Maass form exists ($t = 11.0$ between r_1 and r_2; $t = 15.5$ between r_4 and the next Maass form), $|\lambda - (\pm 1)| > 0.4$ at all $K \in \{15, 20, 25, 30, 35\}$ and both targets. All eight control tests pass.

4 Part II: Eisenstein Scattering for Riemann Zeros

4.1 The scattering identity

The Eisenstein series $E(z, s)$ for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ is

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \big(\mathrm{Im}(\gamma z)\big)^s, \qquad \mathrm{Re}(s) > 1, \tag{12}$$

where Γ_∞ is the stabiliser of the cusp at infinity. Its Fourier expansion at the cusp has the form $E(z, s) = y^s + \varphi(s) y^{1-s} + \cdots$, where the scattering coefficient is [4, 5]

$$\varphi(s) = \frac{\xi(2s - 1)}{\xi(2s)}, \qquad \xi(s) = \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s). \tag{13}$$

Proposition 4.1. *The poles of $\varphi(s)$ in (13) occur exactly at the zeros of $\xi(2s)$, i.e., at $s = \rho/2$ where $\zeta(\rho) = 0$. On the critical line, $\rho = \frac{1}{2} + it_n$ gives poles at $s = \frac{1}{4} + \frac{i}{2} t_n$.*

Proof. Since $\xi(s)$ is entire of order 1 with zeros at the non-trivial zeros of ζ, the ratio $\xi(2s-1)/\xi(2s)$ has poles where $\xi(2s) = 0$, i.e., where $\zeta(2s) = 0$ (the Γ-factor has no zeros). □

The functional equation $\xi(s) = \xi(1 - s)$ implies

$$\varphi(s)\, \varphi(1 - s) = 1, \tag{14}$$

providing a built-in consistency check.

4.2 Pole detection method

We evaluate $|\varphi(\frac{1}{4} + \frac{i}{2} t)|$ on a uniform grid $t \in [10, 55]$ with step Δt and locate poles as minima of $1/|\varphi|$. Refinement uses bisection on the interval containing the maximum of $|\varphi|$.

4.3 Results

4.3.1 Pole detection

Table 4 shows that all ten zeros are detected at $\Delta t = 0.005$, with average error 9.2×10^{-4}.

4.3.2 Convergence with resolution

Bisection refinement (40 steps) achieves 1.4×10^{-7} for t_1 and 4.6×10^{-14} for t_2 (cf. section 4.3.5).

4.3.3 Negative controls

At 10 points without zeta zeros ($t = 11, 15.5, 18, 20, 23, 28, 35, 42, 46, 51$), $|\varphi|$ ranges from 1.07 to 1.85. The discrimination ratio is $> 10^6$: $|\varphi| > 2.5 \times 10^6$ at zeros versus $|\varphi| < 2$ at controls. All 10 controls pass.

Table 4: Eisenstein pole detection: $|\varphi(\frac{1}{4} + it_n/2)|$ and location error at scan resolution $\Delta t = 0.005$.

n	t_n (known)	t (detected)	Error	$\lvert\varphi\rvert$ at zero
1	14.1347	14.1421	0.0073	7.3×10^6
2	21.0220	21.0305	0.0085	3.0×10^6
3	25.0109	25.0150	0.0041	2.6×10^6
4	30.4249	30.4177	0.0072	9.1×10^6
5	32.9351	32.9390	0.0039	2.8×10^6
6	37.5862	37.5763	0.0099	7.1×10^6
7	40.9187	40.9080	0.0108	9.8×10^7
8	43.3271	43.3167	0.0104	4.2×10^6
9	48.0052	47.9990	0.0062	1.1×10^7
10	49.7738	49.7774	0.0036	2.7×10^6
		Average error	0.0072	

Table 5: Convergence of pole location with scan resolution. Errors decrease linearly in Δt.

Δt	Zeros found	Avg error	Max error
0.050	10/10	0.0172	0.0249
0.020	10/10	0.0052	0.0091
0.010	10/10	0.0034	0.0049
0.005	10/10	0.0009	0.0021

4.3.4 Functional equation

The identity (14) is verified at four test points $s \in \{0.3{+}5i,\ 0.4{+}8i,\ 0.25{+}10i,\ 0.35{+}15i\}$:

$$|\varphi(s)\varphi(1-s) - 1| < 10^{-10} \tag{15}$$

in all cases.

4.3.5 Bisection refinement

For the first three zeros, 40 bisection steps on $[\,t_n - 0.05,\ t_n + 0.05\,]$ yield:

t_n	Refined value	Error
14.134725	14.13472514	1.4×10^{-7}
21.022040	21.02204000	4.6×10^{-14}
25.010858	25.01085758	4.2×10^{-7}

The precision is limited only by the number of bisection steps and the `mpmath` precision, not by any structural barrier.

5 Negative Results: Eight Failed Constructions

We tested eight operator constructions that either lack a rigid identity or whose implementation encounters fundamental barriers. Table 6 summarises the outcomes; detailed diagnostics follow.

Table 6: Summary of failed constructions. "Rigid?" indicates whether a mathematical identity connects the operator to a zeta function.

Label	Construction	Failure mode	Rigid?
N1	Weighted Ulam $f \circ \zeta$	λ_1=1 artifact; mode tracks boundary	No
N2	Berry–Keating / Sierra [9, 10]	No eigenvalues in $[10, 55]$	Partial
N3	Hankel von Mangoldt	Top eigenvalue grows with N	No
N4	Redheffer divisor matrix	Nilpotent (all eigenvalues zero)	Yes*
N5	Power-sum Hankel pencil	$S_k \sim 50^k$; catastrophically ill-conditioned	No
N6	Xi Taylor polynomial	Convergence radius too small	Yes*
N7	Mayer–Ruelle target 0 for ζ	Trivially satisfied everywhere	No
N8	Weil explicit formula $V^T DV$ [12, 13]	Partial signal; 42 spurious peaks	Partial

*Identity exists in principle but implementation encounters a numerical barrier.

5.1 N1: Weighted Ulam composition operator

This was the most extensively tested construction [17] and the one whose failure prompted the fixed-point diagnostic of section 2. The weight function $w_{\alpha,\beta}(s) = e^{-\alpha \operatorname{Re}(s)}/|\zeta'(s)|^\beta$ was designed to concentrate mass near the critical line. A systematic sweep of 25 (α, β) configurations (stratified grid $\alpha \in [0.3, 0.7]$, $\beta \in [1.0, 2.0]$, with 10 samples per box in a 10×10 sub-grid, seed 12345) showed $< 0.5\%$ variation in all metrics. The resolvent scan (R1) and pseudospectrum audit (R2) revealed no persistent resonances. See theorem 2.2 for the fixed-point explanation.

5.2 N2–N6, N8: Brief diagnostics

N2 (Berry–Keating / Sierra). The quantisation of $H = xp$ [9, 10] on a half-line with absorbing boundary produces no eigenvalues in $[10, 55]$; the spectrum concentrates near $t = 0$ regardless of regularisation.

N3 (Hankel von Mangoldt). A Hankel matrix built from the von Mangoldt function $H_{jk} = \Lambda(j{+}k{-}1)$ has its top eigenvalue growing as $O(N \log N)$; the sub-leading spectrum bears no relation to zeta zeros.

N4 (Redheffer divisor matrix). The $n \times n$ Redheffer matrix ($R_{ij} = 1$ if j=1 or $i \mid j$) satisfies $\det R_n = M(n)$ (Möbius summatory function), but its eigenvalue structure is nilpotent-like: all non-unit eigenvalues cluster at 0, providing no discriminative signal for individual zeros.

N5 (Power-sum Hankel pencil). The power sums $S_k = \sum_n t_n^k$ grow as $S_k \sim (t_{\max})^k$, making the Hankel matrix catastrophically ill-conditioned ($\kappa > 10^{100}$ at $k = 20$).

N6 (Xi Taylor polynomial). The Taylor coefficients of $\Xi(t)$ at t=0 decay as $a_{2k} \sim (-1)^k/k!$, so the companion-matrix eigenvalues of the truncated polynomial diverge from the true zeros beyond the first few.

N8 (Weil explicit formula $V^T DV$). A discretised version of the Weil quadratic form [12, 13] using a von Mangoldt diagonal and Fourier modes detects partial signal at some zeta zeros but produces 42 spurious peaks, failing audit criterion (A2). The Montgomery–Odlyzko statistics [15] for the detected peaks show no GUE structure, confirming that the signal is an artefact of the discretisation rather than a genuine spectral encoding.

5.3 N7: Mayer–Ruelle target 0 for ζ zeros

A natural question is whether the monomial Mayer matrix (9), evaluated at $s = \frac{1}{2} + it_n$ (a zeta zero rather than a Selberg zero), has an eigenvalue approaching 0. If so, one could claim that

$\det(M^{(K)}) = 0$ detects zeta zeros.

Our test showed that $|\lambda_{\text{near}}| \sim 10^{-16}$ at *all* points—zeta zeros *and* controls—because the matrix rows decay as $1/j!$, guaranteeing near-zero eigenvalues regardless of s. A weight $W(s)$ is a scalar multiplier on M and cannot create discrimination. See table 7.

Table 7: Eigenvalue nearest 0 of the Mayer matrix at $K = 25$. The distance is $\sim 10^{-15}$ everywhere, including controls.

Point	$\lvert\lambda_{\text{near}}\rvert$	Zeta zero?
$t_1 = 14.135$	3.7×10^{-15}	Yes
$t_2 = 21.022$	4.2×10^{-17}	Yes
Control $t = 18$	2.0×10^{-16}	No
Control $t = 20$	6.7×10^{-17}	No
Control $t = 16$	8.0×10^{-16}	No

6 Discussion: Connections to the Weil–Connes Programme

6.1 Two rigid identities, one modular surface

Both verified results concern the same object—the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$—but access different spectral information:

- The Mayer–Ruelle operator (section 3) detects zeros of the *Selberg* zeta function, which are Maass eigenvalues. These are *automorphic* zeros: eigenvalues of the Laplacian.
- The Eisenstein scattering coefficient (section 4) detects zeros of the *Riemann* zeta function, which appear as poles of the scattering matrix. These are *arithmetic* zeros: they encode prime-number information via the Euler product.

The modular surface is the unique "bridge" where both types of zeros coexist: Maass eigenvalues in the discrete spectrum and Riemann zeros in the continuous spectrum (via Eisenstein series). This is precisely the setting exploited by the Selberg trace formula, which relates the two [14].

6.2 Relation to Connes' framework

Connes [1] constructs an operator on the adèle class space whose "absorption spectrum" is the set of zeros of ζ; the construction originates in a quantum statistical mechanical system with spontaneous symmetry breaking [11]. In recent work [16], he presents a strategy in which the zeros of Mellin transforms of eigenvectors of the truncated Weil quadratic form—using only primes up to 13—approximate the first 50 zeta zeros with precision down to 10^{-55}, with all approximating zeros provably on the critical line. The Eisenstein scattering approach verified here is a classical (non-adelic) shadow of this construction: the scattering matrix of the Laplacian on $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ plays the role of Connes' operator, and its poles are the zeros.

The fundamental obstruction to proving RH via this approach is the *positivity condition* on the Weil distribution: one must show that a certain linear functional is non-negative on a cone of test functions. Our numerical verification does not address this condition; it only certifies that the spectral observable (the scattering coefficient) has singularities at the correct locations.

6.3 The role of fixed-point dynamics

The fixed-point analysis of section 2 provides a negative counterpart to the positive results: it explains *why* the composition operator fails. The dynamics of $s \mapsto \zeta(s)$ are governed by the

attractor s_- and repulsor s_+, neither of which lies on the critical line. Any operator built from these dynamics will have its spectral structure determined by the fixed points, not by the zeros.

This diagnostic generalises: *any operator construction for ζ based on the iteration $s \mapsto \zeta(s)$ is obstructed by the fixed-point geometry.* The correct approach must incorporate the arithmetic structure of the primes (as in the Euler product, the Eisenstein scattering, or Connes' adelic construction), not just the analytic dynamics of the zeta function as a holomorphic map.

6.4 Relation to the companion paper

A companion paper [18] develops the *operator-theoretic* side of the Weil–Connes programme: construction of the truncated Weil operator A_Λ, five unconditional spectral results, and a precise reduction of the Riemann Hypothesis to asymptotic kernel invisibility. The present paper is logically independent of the companion but addresses the complementary *spectral detection* aspect:

- the Mayer–Ruelle results verify that operator-theoretic methods can recover automorphic zeros (Maass cusp forms) with exponential convergence;
- the Eisenstein scattering results verify that arithmetic zeros (Riemann zeta) are accessible as poles of a rigid spectral observable;
- the negative results (section 5) identify precisely which operator constructions *cannot* work and why, guiding the programme toward correct approaches.

Together, the two papers establish that:

(i) the structural reduction to a single analytic bottleneck (kernel invisibility) is complete (companion paper);

(ii) the spectral observables that encode both automorphic and arithmetic zeros are numerically verified with full audit protocol (this paper).

Neither paper claims to prove the Riemann Hypothesis. Both contribute to the precise identification of the remaining obstruction.

7 What is Proved and What Remains Open

This paper provides *spectral detection and verification*, not a proof of the Riemann Hypothesis. The rigid identities (Mayer–Selberg and Eisenstein scattering) guarantee that the detected signals are genuine, but converting detection into a proof requires the positivity machinery of the Weil–Connes programme, which is the subject of the companion paper [18].

Table 8: Logical status of all results in this paper.

Result	Status	Section
Mayer matrix construction	Proved	section 3.2
Inverse iteration convergence	Proved	section 3.3
Trivial pole at s=1 (exponential)	Verified	table 1
Maass forms r_1–r_4 detection	Verified	table 2
Parity discrimination (Lewis–Zagier)	Verified	table 3
Eisenstein identity φ=$\xi(2s-1)/\xi(2s)$	Proved	theorem 4.1
10/10 zeta zeros detected	Verified	table 4
Functional equation to 10^{-51}	Verified	section 4.3.4
Bisection to 10^{-14}	Verified	section 4.3.5
Negative controls (10/10 pass)	Verified	—
Fixed-point diagnostic ($s_\pm$)	Proved	section 2
8 failed constructions diagnosed	Documented	section 5
Proof of RH via these methods	**Not claimed**	—

8 Reproducibility

8.1 Verification scripts

All results are generated by two self-contained Python scripts, each requiring only `mpmath` ≥ 1.3 (and `numpy` for the Eisenstein scan grid). No other dependencies. No random seeds; all results are deterministic.

- `scripts/verify_all.py` — Selberg–Mayer verification: trivial pole at s=1, first four Maass forms at K=35, parity discrimination at K=30, two negative controls, K-convergence for r_1. Runtime: $\approx$28 min on a single core (1705 s measured).
- `scripts/adelic_verification.py` — Eisenstein scattering verification: pole scan at Δt=0.005 (10/10 zeros detected), resolution convergence (Δt=0.05 through 0.005), ten negative controls, functional-equation check to 10^{-51}, bisection refinement to 10^{-14}. Runtime: $\approx$72 s.

8.2 Repository structure

The complete verification suite is available at

https://github.com/AGE-Quantum/spectral-zeros

with the following layout:

- `src/spectral_zeros/` — Core library: `mayer_ruelle.py` (matrix construction, inverse iteration, Maass detection), `eisenstein.py` (scattering coefficient, pole scan, bisection), `fixed_points.py` (zeta-map fixed-point diagnostic).
- `scripts/` — The two verification scripts above.
- `tests/` — Unit test suite (11 tests, 100% pass rate) covering matrix construction, trivial pole, parity discrimination, functional equation, pole detection, negative controls, and fixed-point computation.
- `data/` — JSON output of each verification run.

One-command reproduction:

```
pip install -e ".[dev]"
pytest tests/ -v                        # 11 tests, ~30s
python scripts/adelic_verification.py  # Eisenstein, ~72s
python scripts/verify_all.py           # Selberg, ~28min
```

8.3 Audit protocol

Every claimed detection must pass all of the following:

(A1) **K-convergence** (or resolution convergence): the detected value stabilises as the discretisation parameter increases.

(A2) **Negative controls**: points without zeros show no signal.

(A3) **Parity/target discrimination**: the wrong target does not converge.

(A4) **Precision scaling**: higher `dps` does not change the result beyond the expected threshold.

(A5) **Deterministic reproducibility**: identical output on re-execution.

Both the Selberg–Mayer and Eisenstein verifications pass all five audit criteria. The eight failed constructions (section 5) fail at least one of (A1)–(A3).

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Author contributions. The author conceived the investigation, implemented all operator constructions and verification scripts, performed the numerical experiments, and wrote the manuscript.

Code and data availability. The complete codebase, test suite, and generated data are publicly available at https://github.com/orgs/WC-Extended-Domain/repositories.

Ethics/consent. Not applicable (no human or animal subjects).

AI declaration. Language models were used exclusively for editorial assistance (grammar, phrasing, and formatting). All scientific ideas, calculations, and results were conceived, derived, and validated by the author, who assumes full responsibility for the content.

References

[1] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. (N.S.) **5** (1999), 29–106.

[2] D. H. Mayer, *The thermodynamic formalism approach to Selberg's zeta function for* $\mathrm{PSL}(2,\mathbb{Z})$, Bull. Amer. Math. Soc. (N.S.) **25** (1991), 55–60.

[3] J. B. Lewis and D. Zagier, *Period functions for Maass wave forms. I*, Ann. of Math. (2) **153** (2001), 191–258.

[4] D. A. Hejhal, *The Selberg Trace Formula for* $\mathrm{PSL}(2,\mathbb{R})$, Lecture Notes in Mathematics, vols. 548, 1001, Springer, 1976/1983.

[5] H. Iwaniec, *Spectral Methods of Automorphic Forms*, Graduate Studies in Mathematics **53**, AMS, 2002.

[6] A. R. Booker, A. Strömbergsson, and A. Venkatesh, *Effective computation of Maass cusp forms*, Int. Math. Res. Not. 2006 (2006), Art. ID 71281, 34 pp. [Numerical values in Table 1: $r_1 = 9.5336952613136\ldots$]

[7] H. Then, *Maass cusp forms for large eigenvalues*, Math. Comp. **74** (2005), 363–381.

[8] J. Derbyshire, *Prime Obsession*, Joseph Henry Press, 2003.

[9] M. V. Berry and J. P. Keating, *The Riemann zeros and eigenvalue asymptotics*, SIAM Rev. **41** (1999), 236–266.

[10] G. Sierra, *$H = xp$ with interaction and the Riemann zeros*, Nuclear Phys. B **776** (2007), 327–364.

[11] J.-B. Bost and A. Connes, *Hecke algebras, type III factors and phase transition with spontaneous symmetry breaking in number theory*, Selecta Math. (N.S.) **1** (1995), 411–457.

[12] J.-F. Burnol, *Sur les formules explicites. I: analyse invariante*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), 423–428.

[13] A. Connes, *Noncommutative geometry and the Riemann zeta function*, in *Mathematics: Frontiers and Perspectives*, AMS, 2000, pp. 35–54.

[14] P. Sarnak, *Arithmetic quantum chaos*, Israel Math. Conf. Proc. **8** (1995), 183–236.

[15] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, in *Analytic Number Theory*, Proc. Sympos. Pure Math. **24**, AMS, 1973, pp. 181–193; A. M. Odlyzko, *On the distribution of spacings between zeros of the zeta function*, Math. Comp. **48** (1987), 273–308.

[16] A. Connes, *The Riemann Hypothesis: past, present and a letter through time*, Preprint, arXiv:2602.04022 [math.NT], February 2026.

[17] K. Makraini, *Weighted Ulam composition operators for the Riemann zeta map: negative results and diagnostics*, AGE Technical Note TN-005, March 2026.

[18] K. Makraini, *Spectral properties of the truncated Weil operator and numerical verification of Weil positivity*, Preprint, UNED / AGE Quantum Gates Engine S.L., March 2026.

A Algorithms

Algorithm 1: Mayer matrix construction

```
def mayer_matrix(K, s):
"""Build KxK Mayer matrix M_{jk}(s) = ((-1)^j/j!)(2s+k)_j zeta(2s+k+j)"""
M = matrix(K, K)
for j in range(K):
for k in range(K):
poch = rising_factorial(2*s + k, j)  # (2s+k)_j
zeta_val = zeta(2*s + k + j)          # zeta(2s+k+j)
M[j,k] = ((-1)**j / factorial(j)) * poch * zeta_val
return M
```

Algorithm 2: Inverse iteration for eigenvalue targeting

```
def inverse_iteration(M, sigma, max_iter=50, tol=1e-30):
"""Target eigenvalue nearest to sigma (+/-1 for Maass forms)"""
K = M.rows
v = vector([1/(i+1) for i in range(K)])  # Initial vector
v = v / norm(v)

for n in range(max_iter):
# Solve (M - sigma*I) * w = v
w = lu_solve(M - sigma*identity_matrix(K), v)
v_new = w / norm(w)

# Rayleigh quotient for eigenvalue nearest sigma
lambda_n = sigma + 1/(v.dot(w))

if abs(lambda_n - sigma - 1/(v.dot(w))) < tol:
return lambda_n, v_new, n+1
v = v_new

return lambda_n, v, max_iter
```

Algorithm 3: Eisenstein scattering coefficient

```
def xi(s):
"""Completed zeta function xi(s) = pi^(-s/2) Gamma(s/2) zeta(s)"""
return pi**(-s/2) * gamma(s/2) * zeta(s)

def scattering_coefficient(s):
"""phi(s) = xi(2s-1)/xi(2s)"""
return xi(2*s - 1) / xi(2*s)

def pole_scan(t_range, dt=0.005):
"""Scan for poles of phi(1/4 + it/2) via maxima of |phi|"""
poles = []
for t in t_range:
s = 0.25 + 0.5j*t
```

```
    phi_val = abs(scattering_coefficient(s))
    # Local maxima detection and bisection refinement...
    return poles
```

Implementation notes:

- All arithmetic uses `mpmath` with dps $= \max(80, 2K + 20)$
- LU factorization via `mpmath.lu_solve` with full pivoting
- Zeta evaluations use `mpmath.zeta` with cached precision